40

QUANTUM MECHANICS

40.1. **IDENTIFY:** Using the momentum of the free electron, we can calculate k and ω and use these to express its wave function.

SET UP: $\Psi(x,t) = Ae^{ikx}e^{-i\omega t}$, $k = p/\hbar$, and $\omega = \hbar k^2/2m$.

EXECUTE: $k = \dfrac{p}{\hbar} = -\dfrac{4.50\times10^{-24}\ \text{kg}\cdot\text{m/s}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = -4.27\times10^{10}\ \text{m}^{-1}$.

$\omega = \dfrac{\hbar k^2}{2m} = \dfrac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})(4.27\times10^{10}\ \text{m}^{-1})^2}{2(9.108\times10^{-31}\ \text{kg})} = 1.05\times10^{17}\ \text{s}^{-1}$.

$\Psi(x,t) = Ae^{-i[4.27\times10^{10}\ \text{m}^{-1})x}e^{-i[1.05\times10^{17}\ \text{s}^{-1}]t}$.

EVALUATE: The wave function depends on position and time.

40.2. **IDENTIFY:** Using the known wave function for the particle, we want to find where its probability function is a maximum.

SET UP: $|\Psi(x,t)|^2 = |A|^2[e^{ikx}e^{-i\omega t} - e^{2ikx}e^{-4i\omega t}][e^{-ikx}e^{+i\omega t} - e^{-2ikx}e^{+4i\omega t}]$.

$|\Psi(x,t)|^2 = |A|^2(2-[e^{-i(kx-3\omega t)} + e^{+i(kx-3\omega t)}]) = 2|A|^2(1-\cos(kx-3\omega t))$.

EXECUTE: **(a)** For $t=0$, $|\Psi(x,t)|^2 = 2|A|^2(1-\cos(kx))$. $|\Psi(x,t)|^2$ is a maximum when $\cos(kx) = -1$ and this happens when $kx = (2n+1)\pi$, $n = 0,1,\ldots$. $|\Psi(x,t)|^2$ is a maximum for $x = \dfrac{\pi}{k}, \dfrac{3\pi}{k}$, etc.

(b) $t = \dfrac{2\pi}{\omega}$ and $3\omega t = 6\pi$. $|\Psi(x,t)|^2 = 2|A|^2(1-\cos(kx-6\pi))$. Maximum for $kx - 6\pi = \pi, 3\pi, \ldots$, which gives maxima when $x = \dfrac{7\pi}{k}, \dfrac{9\pi}{k}$.

(c) From the results for parts (a) and (b), $v_{\text{av}} = \dfrac{7\pi/k - \pi/k}{2\pi/\omega} = \dfrac{3\omega}{k}$. $v_{\text{av}} = \dfrac{\omega_2-\omega_1}{k_2-k_1}$ with $\omega_2 = 4\omega$, $\omega_1 = \omega$, $k_2 = 2k$ and $k_1 = k$ gives $v_{\text{av}} = \dfrac{3\omega}{k}$.

EVALUATE: The expressions in part (c) agree.

40.3. **IDENTIFY:** Use the wave function from Example 40.1.

SET UP: $|\Psi(x,t)|^2 = 2|A|^2\{1+\cos[(k_2-k_1)x-(\omega_2-\omega_1)t]\}$. $k_2 = 3k_1 = 3k$. $\omega = \dfrac{\hbar k^2}{2m}$, so $\omega_2 = 9\omega_1 = 9\omega$.

$|\Psi(x,t)|^2 = 2|A|^2\{1+\cos(2kx-8\omega t)\}$.

EXECUTE: **(a)** At $t = 2\pi/\omega$, $|\Psi(x,t)|^2 = 2|A|^2\{1+\cos(2kx-16\pi)\}$. $|\Psi(x,t)|^2$ is maximum for $\cos(2kx-16\pi) = 1$. This happens for $2kx - 16\pi = 0, 2\pi, \ldots$. Smallest positive x where $|\Psi(x,t)|^2$ is a maximum is $x = \dfrac{8\pi}{k}$.

(b) From the result of part (a), $v_{\text{av}} = \dfrac{8\pi/k}{2\pi/\omega} = \dfrac{4\omega}{k}$. $v_{\text{av}} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{8\omega}{2k} = \dfrac{4\omega}{k}$.

EVALUATE: The two expressions agree.

40.4. **IDENTIFY:** We have a free particle, described in Example 40.1.

SET UP and **EXECUTE:** $v_{\text{av}} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{\hbar}{2m}\dfrac{(k_2^2 - k_1^2)}{k_2 - k_1} = \dfrac{\hbar}{2m}\dfrac{(k_2 + k_1)(k_2 - k_1)}{k_2 - k_1} = \dfrac{\hbar}{2m}(k_2 + k_1) = \dfrac{p_{\text{av}}}{m}$.

EVALUATE: This is the same as the classical physics result, $v = p/m = mv/m = v$.

40.5. **IDENTIFY** and **SET UP:** $\psi(x) = A\sin kx$. The position probability density is given by $|\psi(x)|^2 = A^2\sin^2 kx$.

EXECUTE: **(a)** The probability is highest where $\sin kx = 1$ so $kx = 2\pi x/\lambda = n\pi/2$, $n = 1, 3, 5, \ldots$

$x = n\lambda/4$, $n = 1, 3, 5, \ldots$ so $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$

(b) The probability of finding the particle is zero where $|\psi|^2 = 0$, which occurs where $\sin kx = 0$ and $kx = 2\pi x/\lambda = n\pi$, $n = 0, 1, 2, \ldots$

$x = n\lambda/2$, $n = 0, 1, 2, \ldots$ so $x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$

EVALUATE: The situation is analogous to a standing wave, with the probability analogous to the square of the amplitude of the standing wave.

40.6. **IDENTIFY** and **SET UP:** $|\Psi|^2 = \Psi^*\Psi$

EXECUTE: $\Psi^* = \psi^*\sin\omega t$, so $|\Psi|^2 = \Psi^*\Psi = \psi^*\psi\sin^2\omega t = |\psi|^2\sin^2\omega t$. $|\Psi|^2$ is not time-independent, so Ψ is not the wavefunction for a stationary state.

EVALUATE: $\Psi = \psi e^{i\omega\phi} = \psi(\cos\omega t + i\sin\omega t)$ is a wavefunction for a stationary state, since for it $|\Psi|^2 = |\psi|^2$, which is time independent.

40.7. **IDENTIFY:** Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$, for some value of E.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E_1\psi_1$ and $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + U\psi_2 = E_2\psi_2$

EXECUTE: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = BE_1\psi_1 + CE_2\psi_2$. If ψ were a solution with energy E, then $BE_1\psi_1 + CE_2\psi_2 = BE\psi_1 + CE\psi_2$ or $B(E_1 - E)\psi_1 = C(E - E_2)\psi_2$. This would mean that ψ_1 is a constant multiple of ψ_2, and ψ_1 and ψ_2 would be wave functions with the same energy. However, $E_1 \neq E_2$, so this is not possible, and ψ cannot be a solution to Eq. (40.23).

EVALUATE: ψ is a solution if $E_1 = E_2$; see Exercise 40.9.

40.8. **IDENTIFY:** Apply the Heisenberg Uncertainty Principle in the form $\Delta x\Delta p_x \geq \hbar/2$.

SET UP: The uncertainty in the particle position is proportional to the width of $\psi(x)$.

EXECUTE: The width of $\psi(x)$ is inversely proportional to $\sqrt{\alpha}$. This can be seen by either plotting the function for different values of α or by finding the full width at half-maximum. The particle's uncertainty in position decreases with increasing α.

(b) Since the uncertainty in position decreases, the uncertainty in momentum must increase.

EVALUATE: As α increases, the function $A(k)$ in Eq. (40.19) must become broader.

40.9. **IDENTIFY:** Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$.

SET UP: ψ_1 and ψ_2 are solutions with energy E means that $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E\psi_1$ and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + U\psi_2 = E\psi_2.$$

EXECUTE: Eq. (40.23): $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2}+U\psi=E\psi.$ Let $\psi=A\psi_1+B\psi_2$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}(A\psi_1+B\psi_2)+U(A\psi_1+B\psi_2)=E(A\psi_1+B\psi_2)$$

$\Rightarrow A\left(-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2}+U\psi_1-E\psi_1\right)+B\left(-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2}+U\psi_2-E\psi_2\right)=0.$ But each of ψ_1 and ψ_2 satisfy Schrödinger's equation separately so the equation still holds true, for any A or B.

EVALUATE: If ψ_1 and ψ_2 are solutions of the Schrodinger equation for different energies, then $\psi=B\psi_1+C\psi_2$ is not a solution (Exercise 40.7).

40.10. IDENTIFY: To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\,dV$ is the probability that the particle is found in volume dV. Since the particle must be *somewhere*, ψ must have the property that $\int|\psi|^2 dV=1$ when the integral is taken over all space.

EXECUTE: **(a)** For normalization of the one-dimensional wave function, we have

$$1=\int_{-\infty}^{\infty}|\psi|^2dx=\int_{-\infty}^{0}(Ae^{bx})^2dx+\int_0^{\infty}(Ae^{-bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx+\int_0^{\infty}A^2e^{-2bx}dx.$$

$$1=A^2\left\{\frac{e^{2bx}}{2b}\bigg|_{-\infty}^{0}+\frac{e^{-2bx}}{-2b}\bigg|_0^{\infty}\right\}=\frac{A^2}{b},\ \text{which gives}\ A=\sqrt{b}=\sqrt{2.00\ \text{m}^{-1}}=1.41\ \text{m}^{-1/2}$$

(b) The graph of the wavefunction versus x is given in Figure 40.10.

(c) (i) $P=\int_{-0.500\ \text{m}}^{+5.00\ \text{m}}|\psi|^2dx=2\int_0^{+5.00\ \text{m}}A^2e^{-2bx}dx,$ where we have used the fact that the wave function is an even function of x. Evaluating the integral gives

$$P=\frac{-A^2}{b}(e^{-2b(0.500\ \text{m})}-1)=\frac{-(2.00\ \text{m}^{-1})}{2.00\ \text{m}^{-1}}(e^{-2.00}-1)=0.865$$

There is a little more than an 86% probability that the particle will be found within 50 cm of the origin.

(ii) $P=\int_{-\infty}^{0}(Ae^{bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx=\dfrac{A^2}{2b}=\dfrac{2.00\ \text{m}^{-1}}{2(2.00\ \text{m}^{-1})}=\dfrac{1}{2}=0.500$

There is a 50-50 chance that the particle will be found to the left of the origin, which agrees with the fact that the wave function is symmetric about the y-axis.

(iii) $P=\int_{0.500\ \text{m}}^{1.00\ \text{m}}A^2e^{-2bx}dx$

$$=\frac{A^2}{-2b}(e^{-2(2.00\ \text{m}^{-1})(1.00\ \text{m})}-e^{-2(2.00\ \text{m}^{-1})(0.500\ \text{m})})=-\frac{1}{2}(e^{-4}-e^{-2})=0.0585$$

EVALUATE: There is little chance of finding the particle in regions where the wave function is small.

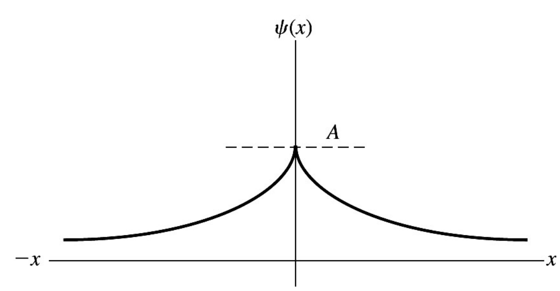

Figure 40.10

40.11. **IDENTIFY** and **SET UP:** The energy levels for a particle in a box are given by $E_n = \dfrac{n^2h^2}{8mL^2}$.

EXECUTE: **(a)** The lowest level is for $n = 1$, and $E_1 = \dfrac{(1)(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(0.20\ \text{kg})(1.3\ \text{m})^2} = 1.6\times10^{-67}\ \text{J}$.

(b) $E = \dfrac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2E}{m}} = \sqrt{\dfrac{2(1.2\times10^{-67}\ \text{J})}{0.20\ \text{kg}}} = 1.3\times10^{-33}\ \text{m/s}$. If the ball has this speed the time it would take it to travel from one side of the table to the other is

$$t = \frac{1.3\ \text{m}}{1.3\times10^{-33}\ \text{m/s}} = 1.0\times10^{33}\ \text{s}.$$

(c) $E_1 = \dfrac{h^2}{8mL^2}$, $E_2 = 4E_1$, so $\Delta E = E_2 - E_1 = 3E_1 = 3(1.6\times10^{-67}\ \text{J}) = 4.9\times10^{-67}\ \text{J}$.

(d) **EVALUATE:** No, quantum mechanical effects are not important for the game of billiards. The discrete, quantized nature of the energy levels is completely unobservable.

40.12. **IDENTIFY:** Solve Eq. (40.31) for L.

SET UP: The ground state has $n = 1$.

EXECUTE: $L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(1.673\times10^{-27}\ \text{kg})(5.0\times10^6\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 6.4\times10^{-15}\ \text{m}$

EVALUATE: The value of L we calculated is on the order of the diameter of a nucleus.

40.13. **IDENTIFY:** An electron in the lowest energy state in this box must have the same energy as it would in the ground state of hydrogen.

SET UP: The energy of the n^{th} level of an electron in a box is $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: An electron in the ground state of hydrogen has an energy of $-13.6\ \text{eV}$, so find the width corresponding to an energy of $E_1 = 13.6\ \text{eV}$. Solving for L gives

$$L = \frac{h}{\sqrt{8mE_1}} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(9.11\times10^{-31}\ \text{kg})(13.6\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 1.66\times10^{-10}\ \text{m}.$$

EVALUATE: This width is of the same order of magnitude as the diameter of a Bohr atom with the electron in the K shell.

40.14. **IDENTIFY** and **SET UP:** The energy of a photon is $E = hf = h\dfrac{c}{\lambda}$. The energy levels of a particle in a box are given by Eq. (40.31).

EXECUTE: **(a)** $E = (6.63\times10^{-34}\ \text{J}\cdot\text{s})\dfrac{(3.00\times10^8\ \text{m/s})}{(122\times10^{-9}\ \text{m})} = 1.63\times10^{-18}\ \text{J}$. $\Delta E = \dfrac{h^2}{8mL^2}(n_1^2 - n_2^2)$.

$$L = \sqrt{\frac{h^2(n_1^2 - n_2^2)}{8m\Delta E}} = \sqrt{\frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2(2^2 - 1^2)}{8(9.11\times10^{-31}\ \text{kg})(1.63\times10^{-18}\ \text{J})}} = 3.33\times10^{-10}\ \text{m}.$$

(b) The ground state energy for an electron in a box of the calculated dimensions is

$E = \dfrac{h^2}{8mL^2} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(3.33\times10^{-10}\ \text{m})^2} = 5.43\times10^{-19}\ \text{J} = 3.40\ \text{eV}$ (one-third of the original photon energy), which does not correspond to the $-13.6\ \text{eV}$ ground state energy of the hydrogen atom.

EVALUATE: **(c)** Note that the energy levels for a particle in a box are proportional to n^2, whereas the energy levels for the hydrogen atom are proportional to $-\dfrac{1}{n^2}$. A one-dimensional box is not a good model for a hydrogen atom.

40.15. **IDENTIFY** and **SET UP:** Eq. (40.31) gives the energy levels. Use this to obtain an expression for $E_2 - E_1$ and use the value given for this energy difference to solve for L.

EXECUTE: Ground state energy is $E_1 = \dfrac{h^2}{8mL^2}$; first excited state energy is $E_2 = \dfrac{4h^2}{8mL^2}$. The energy separation between these two levels is $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$. This gives $L = h\sqrt{\dfrac{3}{8m\Delta E}} =$

$$L = 6.626\times 10^{-34}\ \text{J}\cdot\text{s}\sqrt{\frac{3}{8(9.109\times 10^{-31}\ \text{kg})(3.0\ \text{eV})(1.602\times 10^{-19}\ \text{J/1 eV})}} = 6.1\times 10^{-10}\ \text{m} = 0.61\ \text{nm}.$$

EVALUATE: This energy difference is typical for an atom and L is comparable to the size of an atom.

40.16. **IDENTIFY:** The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and **EXECUTE:** The second excited state energy is $E_3 = \dfrac{9\pi^2\hbar^2}{2mL^2}$. The ground state energy is $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$. $E_1 = 1.00$ eV, so $E_3 = 9.00$ eV. For the transition $\Delta E = \dfrac{4\pi^2\hbar^2}{mL^2}$. $\dfrac{hc}{\lambda} = \Delta E$.

$$\lambda = \frac{hc}{\Delta E} = \frac{(4.136\times 10^{-15}\ \text{eV}\cdot\text{s})(2.998\times 10^{8}\ \text{m/s})}{8.00\ \text{eV}} = 1.55\times 10^{-7}\ \text{m} = 155\ \text{nm}.$$

EVALUATE: This wavelength is much shorter than those of visible light.

40.17. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: We must substitute the equation $\Psi(x,t) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{n\pi x}{L}\right)e^{-iE_n t/\hbar}$ into the one-dimensional Schrödinger equation $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$.

EXECUTE: Taking the second derivative of $\Psi(x,t)$ with respect to x gives $\dfrac{d^2\Psi(x,t)}{dx^2} = -\left(\dfrac{n\pi}{L}\right)^2\Psi(x,t)$.

Substituting this result into $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$, we get $\dfrac{\hbar^2}{2m}\left(\dfrac{n\pi}{L}\right)^2\Psi(x,t) = E\Psi(x,t)$

which gives $E_n = \dfrac{\hbar^2}{2m}\left(\dfrac{n\pi}{L}\right)^2$, the energies of a particle in a box.

EVALUATE: Since this process gives us the energies of a particle in a box, the given wave function is a solution to the Schrödinger equation

40.18. **IDENTIFY:** Find x where ψ_1 is zero and where it is a maximum.

SET UP: $\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$.

EXECUTE: **(a)** The wave function for $n = 1$ vanishes only at $x = 0$ and $x = L$ in the range $0 \le x \le L$.
(b) In the range for x, the sine term is a maximum only at the middle of the box, $x = L/2$.

EVALUATE: **(c)** The answers to parts (a) and (b) are consistent with the figure.

40.19. **IDENTIFY** and **SET UP:** For the $n = 2$ first excited state the normalized wave function is given by Eq. (40.35). $\psi_2(x) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right)$. $|\psi_2(x)|^2\,dx = \dfrac{2}{L}\sin^2\left(\dfrac{2\pi x}{L}\right)dx$. Examine $|\psi_2(x)|^2\,dx$ and find where it is zero and where it is maximum.

EXECUTE: **(a)** $|\psi_2|^2\,dx = 0$ implies $\sin\left(\dfrac{2\pi x}{L}\right) = 0$

$\dfrac{2\pi x}{L} = m\pi$, $m = 0, 1, 2, \ldots$; $x = m(L/2)$

For $m = 0$, $x = 0$; for $m = 1$, $x = L/2$; for $m = 2$, $x = L$

The probability of finding the particle is zero at $x = 0$, $L/2$, and L.

(b) $|\psi_2|^2\,dx$ is maximum when $\sin\left(\dfrac{2\pi x}{L}\right)=\pm 1$

$\dfrac{2\pi x}{L}=m(\pi/2),\ m=1,\ 3,\ 5,\ \ldots\ ;\ x=m(L/4)$

For $m=1,\ x=L/4$; for $m=3,\ x=3L/4$

The probability of finding the particle is largest at $x=L/4$ and $3L/4$.

(c) EVALUATE: The answers to part (a) correspond to the zeros of $|\psi|^2$ shown in Figure 40.12 in the textbook and the answers to part (b) correspond to the two values of x where $|\psi|^2$ in the figure is maximum.

40.20. **IDENTIFY:** Evaluate $\dfrac{d^2\psi}{dx^2}$ and see if Eq. (40.25) is satisfied. $\psi(x)$ must be zero at the walls, where $U\to\infty$.

SET UP: $\dfrac{d}{dx}\sin kx=k\cos kx.\ \dfrac{d}{dx}\cos kx=-k\sin kx.$

EXECUTE: **(a)** $\dfrac{d^2\psi}{dx^2}=-k^2\psi$, and for ψ to be a solution of Eq. (40.25), $k^2=E\dfrac{2m}{\hbar^2}$.

(b) The wave function must vanish at the rigid walls; the given function will vanish at $x=0$ for any k, but to vanish at $x=L,\ kL=n\pi$ for integer n.

EVALUATE: From Eq. (40.31), $E_n=\dfrac{n^2\pi^2\hbar^2}{2mL^2}$, so $k_n=\dfrac{n\pi}{L}$ and $\psi=A\sin kx$ is the same as ψ_n in Eq. (40.32), except for a different symbol for the normalization constant

40.21. **(a) IDENTIFY** and **SET UP:** $\psi=A\cos kx$. Calculate $d\psi^2/dx^2$ and substitute into Eq. (40.25) to see if this equation is satisfied.

EXECUTE: Eq. (40.25): $-\dfrac{h^2}{8\pi^2 m}\dfrac{d^2\psi}{dx^2}=E\psi$

$$\frac{d\psi}{dx}=A(-k\sin kx)=-Ak\sin kx$$

$$\frac{d^2\psi}{dx^2}=-Ak(k\cos kx)=-Ak^2\cos kx$$

Thus Eq. (40.25) requires $-\dfrac{h^2}{8\pi^2 m}(-Ak^2\cos kx)=E(A\cos kx)$.

This says $\dfrac{h^2k^2}{8\pi^2 m}=E;\ k=\dfrac{\sqrt{2mE}}{(h/2\pi)}=\dfrac{\sqrt{2mE}}{\hbar}$

$\psi=A\cos kx$ is a solution to Eq. (40.25) if $k=\dfrac{\sqrt{2mE}}{\hbar}$.

(b) EVALUATE: The wave function for a particle in a box with rigid walls at $x=0$ and $x=L$ must satisfy the boundary conditions $\psi=0$ at $x=0$ and $\psi=0$ at $x=L$. $\psi(0)=A\cos 0=A$, since $\cos 0=1$. Thus ψ is not 0 at $x=0$ and this wave function isn't acceptable because it doesn't satisfy the required boundary condition, even though it is a solution to the Schrödinger equation.

40.22. **IDENTIFY:** The energy levels are given by Eq. (40.31). The wavelength λ of the photon absorbed in an atomic transition is related to the transition energy ΔE by $\lambda=\dfrac{hc}{\Delta E}$.

SET UP: For the ground state $n=1$ and for the third excited state $n=4$.

EXECUTE: **(a)** The third excited state is $n=4$, so

$$\Delta E=(4^2-1)\frac{h^2}{8mL^2}=\frac{15(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(0.125\times10^{-9}\ \text{m})^2}=5.78\times10^{-17}\ \text{J}=361\ \text{eV}.$$

(b) $\lambda = \dfrac{hc}{\Delta E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.0\times10^{8}\ \text{m/s})}{5.78\times10^{-17}\ \text{J}} = 3.44\ \text{nm}$

EVALUATE: This photon is an x ray. As the width of the box increases the transition energy for this transition decreases and the wavelength of the photon increases.

40.23. **IDENTIFY** and **SET UP:** $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mE}}$. The energy of the electron in level n is given by Eq. (40.31).

EXECUTE: **(a)** $E_1 = \dfrac{h^2}{8mL^2} \Rightarrow \lambda_1 = \dfrac{h}{\sqrt{2mh^2/8mL^2}} = 2L = 2(3.0\times10^{-10}\ \text{m}) = 6.0\times10^{-10}\ \text{m}$. The wavelength is twice the width of the box. $p_1 = \dfrac{h}{\lambda_1} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})}{6.0\times10^{-10}\ \text{m}} = 1.1\times10^{-24}\ \text{kg}\cdot\text{m/s}$.

(b) $E_2 = \dfrac{4h^2}{8mL^2} \Rightarrow \lambda_2 = L = 3.0\times10^{-10}\ \text{m}$. The wavelength is the same as the width of the box.

$p_2 = \dfrac{h}{\lambda_2} = 2p_1 = 2.2\times10^{-24}\ \text{kg}\cdot\text{m/s}$.

(c) $E_3 = \dfrac{9h^2}{8mL^2} \Rightarrow \lambda_3 = \dfrac{2}{3}L = 2.0\times10^{-10}\ \text{m}$. The wavelength is two-thirds the width of the box.

$p_3 = 3p_1 = 3.3\times10^{-24}\ \text{kg}\cdot\text{m/s}$.

EVALUATE: In each case the wavelength is an integer multiple of $\lambda/2$. In the n^{th} state, $p_n = np_1$.

40.24. **IDENTIFY:** To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\ dV$ is the probability that the particle is found in volume dV. Since the particle must be *somewhere*, ψ must have the property that $\int |\psi|^2 dV = 1$ when the integral is taken over all space.

EXECUTE: **(a)** In one dimension, as we have here, the integral discussed above is of the form $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$.

(b) Using the result from part (a), we have $\int_{-\infty}^{\infty} (e^{ax})^2\, dx = \int_{-\infty}^{\infty} e^{2ax} dx = \left.\dfrac{e^{2ax}}{2a}\right|_{-\infty}^{\infty} = \infty$. Hence this wave function cannot be normalized and therefore cannot be a valid wave function.

(c) We only need to integrate this wave function of 0 to ∞ because it is zero for $x < 0$. For normalization we have $1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_0^{\infty} (Ae^{-bx})^2\, dx = \int_0^{\infty} A^2 e^{-2bx} dx = \left.\dfrac{A^2 e^{-2bx}}{-2b}\right|_0^{\infty} = \dfrac{A^2}{2b}$, which gives $\dfrac{A^2}{2b} = 1$, so $A = \sqrt{2b}$.

EVALUATE: If b were negative, the given wave function could not be normalized, so it would not be allowable.

40.25. **IDENTIFY:** Compare $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ to $E\psi$ and see if there is a value of k for which they are equal.

SET UP: $\dfrac{d^2}{dx^2}\sin kx = -k^2 \sin kx$.

EXECUTE: **(a)** Eq. (40.23): $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = E\psi$.

Left-hand side: $\dfrac{-\hbar^2}{2m}\dfrac{d^2}{dx^2}(A\sin kx) + U_0 A\sin kx = \dfrac{\hbar^2 k^2}{2m} A\sin kx + U_0 A\sin kx = \left(\dfrac{\hbar^2 k^2}{2m} + U_0\right)\psi$. But $\dfrac{\hbar^2 k^2}{2m} + U_0 > U_0 > E$ if k is real. But $\dfrac{\hbar^2 k^2}{2m} + U_0$ should equal E. This is not the case, and there is no k for which this $|\psi|^2$ is a solution.

(b) If $E > U_0$, then $\dfrac{\hbar^2 k^2}{2m} + U_0 = E$ is consistent and so $\psi = A\sin kx$ is a solution of Eq. (40.23) for this case.

EVALUATE: For a square-well potential and $E < U_0$, Eq. (40.23) with $U = U_0$ applies outside the well and the wave function has the form of Eq. (40.40).

40.26. **IDENTIFY:** $\lambda = \dfrac{h}{p}$. p is related to E by $E = \dfrac{p^2}{2m} + U$.

SET UP: For $x > L$, $U = U_0$. For $0 < x < L$, $U = 0$.

EXECUTE: For $0 < x < L$, $p = \sqrt{2mE} = \sqrt{2m(3U_0)}$ and $\lambda_{\text{in}} = \dfrac{h}{\sqrt{2m(3U_0)}}$. For $x > L$,

$p = \sqrt{2m(E - U_0)} = \sqrt{2m(2U_0)}$ and $\lambda_{\text{out}} = \dfrac{h}{\sqrt{2m(E-U_0)}} = \dfrac{h}{\sqrt{2m(2U_0)}}$. Thus, the ratio of the

wavelengths is $\dfrac{\lambda_{\text{out}}}{\lambda_{\text{in}}} = \dfrac{\sqrt{2m(3U_0)}}{\sqrt{2m(2U_0)}} = \sqrt{\dfrac{3}{2}}$.

EVALUATE: For $x > L$ some of the energy is potential and the kinetic energy is less than it is for $0 < x < L$, where $U = 0$. Therefore, outside the box p is less and λ is greater than inside the box.

40.27. **IDENTIFY:** Figure 40.15b in the textbook gives values for the bound state energy of a square well for which $U_0 = 6E_{\text{1-IDW}}$.

SET UP: $E_{\text{1-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$.

EXECUTE: $E_1 = 0.625E_{\text{1-IDW}} = 0.625\dfrac{\pi^2\hbar^2}{2mL^2}$; $E_1 = 2.00\text{ eV} = 3.20\times10^{-19}\text{ J}$.

$$L = \pi\hbar\left(\frac{0.625}{2(9.109\times10^{-31}\text{ kg})(3.20\times10^{-19}\text{ J})}\right)^{1/2} = 3.43\times10^{-10}\text{ m}.$$

EVALUATE: As L increases the ground state energy decreases.

40.28. **IDENTIFY:** The energy of the photon is the energy given to the electron.

SET UP: Since $U_0 = 6E_{\text{1-IDW}}$ we can use the result $E_1 = 0.625E_{\text{1-IDW}}$ from Section 40.4. When the electron is outside the well it has potential energy U_0, so the minimum energy that must be given to the electron is $U_0 - E_1 = 5.375E_{\text{1-IDW}}$.

EXECUTE: The maximum wavelength of the photon would be

$$\lambda = \frac{hc}{U_0 - E_1} = \frac{hc}{(5.375)(h^2/8mL^2)} = \frac{8mL^2c}{(5.375)h} = \frac{8(9.11\times10^{-31}\text{ kg})(1.50\times10^{-9}\text{ m})^2(3.00\times10^8\text{ m/s})}{(5.375)(6.63\times10^{-34}\text{ J}\cdot\text{s})}$$
$$= 1.38\times10^{-6}\text{ m}.$$

EVALUATE: This photon is in the infrared. The wavelength of the photon decreases when the width of the well decreases.

40.29. **IDENTIFY:** Calculate $\dfrac{d^2\psi}{dx^2}$ and compare to $-\dfrac{2mE}{\hbar^2}\psi$.

SET UP: $\dfrac{d}{dx}\sin kx = k\cos kx$. $\dfrac{d}{dx}\cos kx = -k\sin kx$.

EXECUTE: Eq. (40.37): $\psi = A\sin\dfrac{\sqrt{2mE}}{\hbar}x + B\cos\dfrac{\sqrt{2mE}}{\hbar}x$.

$\dfrac{d^2\psi}{dx^2} = -A\left(\dfrac{2mE}{\hbar^2}\right)\sin\dfrac{\sqrt{2mE}}{\hbar}x - B\left(\dfrac{2mE}{\hbar^2}\right)\cos\dfrac{\sqrt{2mE}}{\hbar}x = \dfrac{-2mE}{\hbar^2}(\psi)$. This is Eq. (40.38), so this ψ is a solution.

EVALUATE: ψ in Eq. (40.38) is a solution to Eq. (40.37) for any values of the constants A and B.

40.30. **IDENTIFY:** The longest wavelength corresponds to the smallest energy change.

SET UP: The ground level energy level of the infinite well is $E_{\text{1-1DW}} = \dfrac{h^2}{8mL^2}$, and the energy of the photon must be equal to the energy difference between the two shells.

EXECUTE: The 400.0 nm photon must correspond to the $n=1$ to $n=2$ transition. Since $U_0 = 6E_{\text{1-1DW}}$, we have $E_2 = 2.43E_{\text{1-1DW}}$ and $E_1 = 0.625E_{\text{1-1DW}}$. The energy of the photon is equal to the energy difference between the two levels, and $E_{\text{1-1DW}} = \dfrac{h^2}{8mL^2}$, which gives

$E_\gamma = E_2 - E_1 \Rightarrow \dfrac{hc}{\lambda} = (2.43 - 0.625)E_{\text{1-1DW}} = \dfrac{1.805\,h^2}{8mL^2}$. Solving for L gives

$$L = \sqrt{\frac{(1.805)h\lambda}{8mc}} = \sqrt{\frac{(1.805)(6.626\times10^{-34}\ \text{J}\cdot\text{s})(4.00\times10^{-7}\ \text{m})}{8(9.11\times10^{-31}\ \text{kg})(3.00\times10^{8}\ \text{m/s})}} = 4.68\times10^{-10}\ \text{m} = 0.468\ \text{nm}.$$

EVALUATE: This width is approximately half that of a Bohr hydrogen atom.

40.31. **IDENTIFY:** Find the transition energy ΔE and set it equal to the energy of the absorbed photon. Use $E = hc/\lambda$, to find the wavelength of the photon.

SET UP: $U_0 = 6E_{\text{1-1DW}}$, as in Figure 40.15 in the textbook, so $E_1 = 0.625E_{\text{1-1DW}}$ and $E_3 = 5.09E_{\text{1-1DW}}$ with $E_{\text{1-1DW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$. In this problem the particle bound in the well is a proton, so $m = 1.673\times10^{-27}$ kg.

EXECUTE: $E_{\text{1-1DW}} = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(1.673\times10^{-27}\ \text{kg})(4.0\times10^{-15}\ \text{m})^2} = 2.052\times10^{-12}$ J. The transition energy is $\Delta E = E_3 - E_1 = (5.09 - 0.625)E_{\text{1-1DW}} = 4.465E_{\text{1-1DW}}$. $\Delta E = 4.465(2.052\times10^{-12}\ \text{J}) = 9.162\times10^{-12}$ J

The wavelength of the photon that is absorbed is related to the transition energy by $\Delta E = hc/\lambda$, so

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{9.162\times10^{-12}\ \text{J}} = 2.2\times10^{-14}\ \text{m} = 22\ \text{fm}.$$

EVALUATE: The wavelength of the photon is comparable to the size of the box.

40.32. **IDENTIFY:** The tunneling probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$. so

$$T = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{\frac{-2\sqrt{2m(U_0 - E)}}{\hbar}L}.$$

SET UP: $U_0 = 30.0\times10^6$ eV, $L = 2.0\times10^{-15}$ m, $m = 6.64\times10^{-27}$ kg.

EXECUTE: **(a)** $U_0 - E = 1.0\times10^6$ eV $(E = 29.0\times10^6$ eV$)$, $T = 0.090$.

(b) If $U_0 - E = 10.0\times10^6$ eV $(E = 20.0\times10^6$ eV$)$, $T = 0.014$.

EVALUATE: T is less when $U_0 - E$ s 10.0 MeV than when $U_0 - E$ is 1.0 MeV.

40.33. **IDENTIFY:** The tunneling probability is $T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0 - E)}/\hbar}$.

SET UP: $\dfrac{E}{U_0} = \dfrac{6.0\ \text{eV}}{11.0\ \text{eV}}$ and $E - U_0 = 5\ \text{eV} = 8.0\times10^{-19}$ J.

EXECUTE: **(a)** $L = 0.80\times10^{-9}$ m:

$$T = 16\left(\frac{6.0\ \text{eV}}{11.0\ \text{eV}}\right)\left(1 - \frac{6.0\ \text{ev}}{11.0\ \text{eV}}\right)e^{-2(0.80\times10^{-9}\ \text{m})\sqrt{2(9.11\times10^{-31}\ \text{kg})(8.0\times10^{-19}\ \text{J})}/1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 4.4\times10^{-8}.$$

(b) $L = 0.40\times10^{-9}$ m: $T = 4.2\times10^{-4}$.

EVALUATE: The tunneling probability is less when the barrier is wider.

40.34. **IDENTIFY:** The transmission coefficient is $T = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)e^{-2\sqrt{2m(U_0-E)}L/\hbar}$.

SET UP: $E = 5.0$ eV, $L = 0.60\times10^{-9}$ m, and $m = 9.11\times10^{-31}$ kg

EXECUTE: **(a)** $U_0 = 7.0\text{ eV} \Rightarrow T = 5.5\times10^{-4}$.

(b) $U_0 = 9.0\text{ eV} \Rightarrow T = 1.8\times10^{-5}$.

(c) $U_0 = 13.0\text{ eV} \Rightarrow T = 1.1\times10^{-7}$.

EVALUATE: T decreases when the height of the barrier increases.

40.35. **IDENTIFY** and **SET UP:** Use Eq. (39.1), where $K = p^2/2m$ and $E = K + U$.

EXECUTE: $\lambda = h/p = h/\sqrt{2mK}$, so $\lambda\sqrt{K}$ is constant. $\lambda_1\sqrt{K_1} = \lambda_2\sqrt{K_2}$; λ_1 and K_1 are for $x > L$ where $K_1 = 2U_0$ and λ_2 and K_2 are for $0 < x < L$ where $K_2 = E - U_0 = U_0$.

$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{K_2}{K_1}} = \sqrt{\frac{U_0}{2U_0}} = \frac{1}{\sqrt{2}}$$

EVALUATE: When the particle is passing over the barrier its kinetic energy is less and its wavelength is larger.

40.36. **IDENTIFY:** The probability of tunneling depends on the energy of the particle and the width of the barrier.

SET UP: The probability of tunneling is approximately $T = Ge^{-2\kappa L}$, where $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}.$$

EXECUTE: $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right) = 16\dfrac{50.0\text{ eV}}{70.0\text{ eV}}\left(1-\dfrac{50.0\text{ eV}}{70.0\text{ eV}}\right) = 3.27.$

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar} = \frac{\sqrt{2(1.67\times10^{-27}\text{ kg})(70.0\text{ eV}-50.0\text{ eV})(1.60\times10^{-19}\text{ J/eV})}}{(6.63\times10^{-34}\text{ J}\cdot\text{s})/2\pi} = 9.8\times10^{11}\text{ m}^{-1}$$

Solving $T = Ge^{-2\kappa L}$ for L gives

$$L = \frac{1}{2\kappa}\ln(G/T) = \frac{1}{2(9.8\times10^{11}\text{ m}^{-1})}\ln\left(\frac{3.27}{0.0030}\right) = 3.6\times10^{-12}\text{ m} = 3.6\text{ pm}.$$

If the proton were replaced with an electron, the electron's mass is much smaller so L would be larger.

EVALUATE: An electron can tunnel through a much wider barrier than a proton of the same energy.

40.37. **IDENTIFY** and **SET UP:** The probability is $T = Ae^{-2\kappa L}$, with $A = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$.

$E = 32$ eV, $U_0 = 41$ eV, $L = 0.25\times10^{-9}$ m. Calculate T.

EXECUTE: **(a)** $A = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right) = 16\dfrac{32}{41}\left(1-\dfrac{32}{41}\right) = 2.741.$

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}$$

$$\kappa = \frac{\sqrt{2(9.109\times10^{-31}\text{ kg})(41\text{ eV}-32\text{ eV})(1.602\times10^{-19}\text{ J/eV})}}{1.055\times10^{-34}\text{ J}\cdot\text{s}} = 1.536\times10^{10}\text{ m}^{-1}$$

$$T = Ae^{-2\kappa L} = (2.741)e^{-2(1.536\times10^{10}\text{ m}^{-1})(0.25\times10^{-9}\text{ m})} = 2.741e^{-7.68} = 0.0013$$

(b) The only change in the mass m, which appears in κ.

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}$$

$$\kappa = \frac{\sqrt{2(1.673\times10^{-27}\ \text{kg})(41\ \text{eV} - 32\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 6.584\times10^{11}\ \text{m}^{-1}$$

Then $T = Ae^{-2\kappa L} = (2.741)e^{-2(6.584\times10^{11}\ \text{m}^{-1})(0.25\times10^{-9}\ \text{m})} = 2.741e^{-392.2} = 10^{-143}$

EVALUATE: The more massive proton has a much smaller probability of tunneling than the electron does.

40.38. **IDENTIFY:** Calculate $\frac{d^2\psi}{dx^2}$ and insert the result into Eq. (40.44).

SET UP: $\frac{d}{dx}e^{-\delta x^2} = -2\delta x e^{-\delta x^2}$ and $\frac{d^2}{dx^2}e^{-\delta x^2} = (4\delta^2 x^2 - 2\delta)e^{-\delta x^2}$

EXECUTE: Let $\sqrt{mk'}/2\hbar = \delta$, and so $\frac{d\psi}{dx} = -2x\delta\psi$ and $\frac{d^2\psi}{dx^2} = (4x^2\delta^2 - 2\delta)\psi$, and ψ is a solution of Eq. (40.44) if $E = \frac{\hbar^2}{m}\delta = \frac{1}{2}\hbar\sqrt{k'/m} = \frac{1}{2}\hbar\omega.$

EVALUATE: $E = \frac{1}{2}\hbar\omega$ agrees with Eq. (40.46), for $n = 0$.

40.39. **IDENTIFY** and **SET UP:** The energy levels are given by Eq. (40.46), where $\omega = \sqrt{\frac{k'}{m}}.$

EXECUTE: $\omega = \sqrt{\frac{k'}{m}} = \sqrt{\frac{110\ \text{N/m}}{0.250\ \text{kg}}} = 21.0\ \text{rad/s}$

The ground state energy is given by Eq. (40.46):

$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}(1.055\times10^{-34}\ \text{J}\cdot\text{s})(21.0\ \text{rad/s}) = 1.11\times10^{-33}\ \text{J}(1\ \text{eV}/1.602\times10^{-19}\ \text{J}) = 6.93\times10^{-15}\ \text{eV}$

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,\ \ E_{(n+1)} = \left(n + 1 + \frac{1}{2}\right)\hbar\omega$

The energy separation between these adjacent levels is

$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 2(1.11\times10^{-33}\ \text{J}) = 2.22\times10^{-33}\ \text{J} = 1.39\times10^{-14}\ \text{eV}.$

EVALUATE: These energies are extremely small; quantum effects are not important for this oscillator.

40.40. **IDENTIFY:** The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and **EXECUTE:** $\Delta E = \frac{hc}{\lambda} = \frac{(4.136\times10^{-15}\ \text{eV}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{8.65\times10^{-6}\ \text{m}} = 0.1433\ \text{eV}.$ $\Delta E = \hbar\omega.$

$E_0 = \frac{\hbar\omega}{2} = \frac{0.1433\ \text{eV}}{2} = 0.0717\ \text{eV}.$

EVALUATE: The energy of the photon is not equal to the energy of the ground state, but rather it is the energy *difference* between the two states.

40.41. **IDENTIFY:** We can model the molecule as a harmonic oscillator. The energy of the photon is equal to the energy difference between the two levels of the oscillator.

SET UP: The energy of a photon is $E_\gamma = hf = hc/\lambda$, and the energy levels of a harmonic oscillator are given by $E_n = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n + \frac{1}{2}\right)\hbar\omega.$

EXECUTE: **(a)** The photon's energy is $E_\gamma = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^{8}\ \text{m/s})}{5.8\times10^{-6}\ \text{m}} = 0.21\ \text{eV}.$

(b) The transition energy is $\Delta E = E_{n+1} - E_n = \hbar\omega = \hbar\sqrt{\frac{k'}{m}}$, which gives $\frac{2\pi\hbar c}{\lambda} = \hbar\sqrt{\frac{k'}{m}}.$ Solving for k', we get $k' = \frac{4\pi^2 c^2 m}{\lambda^2} = \frac{4\pi^2(3.00\times10^{8}\ \text{m/s})^2(5.6\times10^{-26}\ \text{kg})}{(5.8\times10^{-6}\ \text{m})^2} = 5{,}900\ \text{N/m}.$

EVALUATE: This would be a rather strong spring in the physics lab.

40.42. **IDENTIFY:** The photon energy equals the transition energy for the atom.
SET UP: According to Eq. (40.46), the energy released during the transition between two adjacent levels is twice the ground state energy $E_3 - E_2 = \hbar\omega = 2E_0 = 11.2\text{ eV}$.
EXECUTE: For a photon of energy E,

$$E = hf \Rightarrow \lambda = \frac{c}{f} = \frac{hc}{E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^{8}\text{ m/s})}{(11.2\text{ eV})(1.60\times10^{-19}\text{ J/eV})} = 111\text{ nm}.$$

EVALUATE: This photon is in the ultraviolet.

40.43. **IDENTIFY** and **SET UP:** Use the energies given in Eq. (40.46) to solve for the amplitude A and maximum speed v_{max} of the oscillator. Use these to estimate Δx and Δp_x and compute the uncertainty product $\Delta x \Delta p_x$.

EXECUTE: The total energy of a Newtonian oscillator is given by $E = \frac{1}{2}k'A^2$ where k' is the force constant and A is the amplitude of the oscillator. Set this equal to the energy $E = \left(n+\frac{1}{2}\right)\hbar\omega$ of an excited level that has quantum number n, where $\omega = \sqrt{\frac{k'}{m}}$, and solve for A: $\frac{1}{2}k'A^2 = \left(n+\frac{1}{2}\right)\hbar\omega$.

$A = \sqrt{\frac{(2n+1)\hbar\omega}{k'}}$. The total energy of the Newtonian oscillator can also be written as $E = \frac{1}{2}mv_{\text{max}}^2$. Set this equal to $E = \left(n+\frac{1}{2}\right)\hbar\omega$ and solve for v_{max}: $\frac{1}{2}mv_{\text{max}}^2 = \left(n+\frac{1}{2}\right)\hbar\omega$. $v_{\text{max}} = \sqrt{\frac{(2n+1)\hbar\omega}{m}}$. Thus the maximum linear momentum of the oscillator is $p_{\text{max}} = mv_{\text{max}} = \sqrt{(2n+1)\hbar m\omega}$. Now $A/\sqrt{2}$ represents the uncertainty Δx in position and that $p_{\text{max}}/\sqrt{2}$ is the corresponding uncertainty Δp_x in momentum. Then the uncertainty product is

$$\Delta x \Delta p_x = \left(\frac{1}{\sqrt{2}}\sqrt{\frac{(2n+1)\hbar\omega}{k'}}\right)\left(\frac{1}{\sqrt{2}}\sqrt{(2n+1)\hbar m\omega}\right) = \frac{(2n+1)\hbar\omega}{2}\sqrt{\frac{m}{k'}} = \frac{(2n+1)\hbar\omega}{2}\left(\frac{1}{\omega}\right) = (2n+1)\frac{\hbar}{2}.$$

EVALUATE: For $n = 0$ this gives $\Delta x \Delta p_x = \hbar/2$, in agreement with the result derived in Section 40.5. The uncertainty product $\Delta x \Delta p_x$ increases with n.

40.44. **IDENTIFY:** Compute the ratio specified in the problem.
SET UP: For $n = 0$, $A = \sqrt{\frac{\hbar\omega}{k'}}$. $\omega = \sqrt{\frac{k'}{m}}$.

EXECUTE: **(a)** $\frac{|\psi(A)|^2}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}A^2\right) = \exp\left(-\sqrt{mk'}\frac{\omega}{k'}\right) = e^{-1} = 0.368$. This is consistent with what is shown in Figure 40.27 in the textbook.

(b) $\frac{\left|\psi(2A)^2\right|}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}(2A)^2\right) = \exp\left(-\sqrt{mk'}\,4\frac{\omega}{k'}\right) = e^{-4} = 1.83\times10^{-2}$. This figure cannot be read this precisely, but the qualitative decrease in amplitude with distance is clear.
EVALUATE: The wave function decays exponentially as x increases beyond $x = A$.

40.45. **IDENTIFY:** We model the atomic vibration in the crystal as a harmonic oscillator.
SET UP: The energy levels of a harmonic oscillator are given by $E_n = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n+\frac{1}{2}\right)\hbar\omega$.

EXECUTE: **(a)** The ground state energy of a simple harmonic oscillator is

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{k'}{m}} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})}{2}\sqrt{\frac{12.2\text{ N/m}}{3.82\times10^{-26}\text{ kg}}} = 9.43\times10^{-22}\text{ J} = 5.89\times10^{-3}\text{ eV}$$

(b) $E_4 - E_3 = \hbar\omega = 2E_0 = 0.0118\text{ eV}$, so $\lambda = \frac{hc}{E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^{8}\text{ m/s})}{1.88\times10^{-21}\text{ J}} = 106\ \mu\text{m}$

(c) $E_{n+1}-E_n=\hbar\omega=2E_0=0.0118\text{ eV}$

EVALUATE: These energy differences are much smaller than those due to electron transitions in the hydrogen atom.

40.46. **IDENTIFY:** For a stationary state, $|\Psi|^2$ is time independent.

SET UP: To calculate Ψ^* from Ψ, replace i by $-i$.

EXECUTE: For this wave function, $\Psi^*=\psi_1^* e^{i\omega_1 t}+\psi_2^* e^{i\omega_2 t}$, so

$$|\Psi|^2=\Psi^*\Psi=(\psi_1^* e^{i\omega_1 t}+\psi_2^* e^{i\omega_2 t})(\psi_1 e^{-i\omega_1 t}+\psi_2 e^{-i\omega_2 t})=\psi_1^*\psi_1+\psi_2^*\psi_2+\psi_1^*\psi_2 e^{i(\omega_1-\omega_2)t}+\psi_2^*\psi_1 e^{i(\omega_2-\omega_1)t}.$$

The frequencies ω_1 and ω_2 are given as not being the same, so $|\Psi|^2$ is not time-independent, and Ψ is not the wave function for a stationary state.

EVALUATE: If $\omega_1=\omega_2$, then Ψ is the wave function for a stationary state.

40.47. **IDENTIFY:** We know the wave function of a particle in a box.

SET UP and **EXECUTE:** **(a)** $\Psi(x,t)=\dfrac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar}+\dfrac{1}{\sqrt{2}}\psi_3(x)e^{-iE_3t/\hbar}.$

$$\Psi^*(x,t)=\frac{1}{\sqrt{2}}\psi_1(x)e^{+iE_1t/\hbar}+\frac{1}{\sqrt{2}}\psi_3(x)e^{+iE_3t/\hbar}.$$

$$|\Psi(x,t)|^2=\frac{1}{2}[\psi_1^2+\psi_3^2+\psi_1\psi_3(e^{i(E_3-E_1)t/\hbar}+e^{-i(E_3-E_1)t/\hbar})]=\frac{1}{2}\left[\psi_1^2+\psi_3^2+2\psi_1\psi_3\cos\left(\frac{[E_3-E_1]t}{\hbar}\right)\right].$$

$\psi_1=\sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$. $\psi_3=\sqrt{\dfrac{2}{L}}\sin\left(\dfrac{3\pi x}{L}\right)$. $E_3=\dfrac{9\pi^2\hbar^2}{2mL^2}$ and $E_1=\dfrac{\pi^2\hbar^2}{2mL^2}$, so $E_3-E_1=\dfrac{4\pi^2\hbar^2}{mL^2}$.

$|\Psi(x,t)|^2=\dfrac{1}{L}\left[\sin^2\left(\dfrac{\pi x}{L}\right)+\sin^2\left(\dfrac{3\pi x}{L}\right)+2\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{3\pi x}{L}\right)\cos\left(\dfrac{4\pi^2\hbar t}{mL^2}\right)\right]$. At $x=L/2$,

$\sin\left(\dfrac{\pi x}{L}\right)=\sin\left(\dfrac{\pi}{2}\right)=1$. $\sin\left(\dfrac{3\pi x}{L}\right)=\sin\left(\dfrac{3\pi}{2}\right)=-1$. $|\Psi(x,t)|^2=\dfrac{2}{L}\left[1-\cos\left(\dfrac{4\pi^2\hbar t}{mL^2}\right)\right]$.

(b) $\omega_{\text{osc}}=\dfrac{E_3-E_1}{\hbar}=\dfrac{4\pi^2\hbar}{mL^2}$.

EVALUATE: Note that $\Delta E=\hbar\omega$.

40.48. **IDENTIFY:** Carry out the calculations specified in the problem.

SET UP: A standard integral is $\int_0^\infty e^{-\alpha^2k^2}\cos(kx)dk=\dfrac{\sqrt{\pi}}{2\alpha}e^{-x^2/4\alpha^2}$.

EXECUTE: **(a)** $B(k)=e^{-\alpha^2k^2}$. $B(0)=B_{\max}=1$. $B(k_{\text{h}})=\dfrac{1}{2}=e^{-\alpha^2k_{\text{h}}^2}\Rightarrow\ln(1/2)=-\alpha^2k_{\text{h}}^2$

$\Rightarrow k_{\text{h}}=\dfrac{1}{\alpha}\sqrt{\ln(2)}=w_k$.

(b) $\psi(x)=\int_0^\infty e^{-\alpha^2k^2}\cos kx\,dk=\dfrac{\sqrt{\pi}}{2\alpha}(e^{-x^2/4\alpha^2})$. $\psi(x)$ is a maximum when $x=0$.

(c) $\psi(x_{\text{h}})=\dfrac{\sqrt{\pi}}{4\alpha}$ when $e^{-x_{\text{h}}^2/4\alpha^2}=\dfrac{1}{2}\Rightarrow\dfrac{-x_{\text{h}}^2}{4\alpha^2}=\ln(1/2)\Rightarrow x_{\text{h}}=2\alpha\sqrt{\ln 2}=w_x$

(d) $w_pw_x=\left(\dfrac{hw_k}{2\pi}\right)w_x=\dfrac{h}{2\pi}\left(\dfrac{1}{\alpha}\sqrt{\ln 2}\right)(2\alpha\sqrt{\ln 2})=\dfrac{h}{2\pi}(2\ln 2)=\dfrac{h\ln 2}{\pi}=(2\ln 2)\hbar.$

EVALUATE: The Heisenberg Uncertainty Principle says that $\Delta x\Delta p_x\ge\hbar/2$. If $\Delta x=w_x$ and $\Delta p_x=w_p$, then the uncertainty principle says $w_xw_p\ge\hbar/2$. So our result is consistent with the uncertainty principle since $(2\ln 2)\hbar>\hbar/2$.

40.49. **IDENTIFY:** Evaluate $\psi(x)=\int_0^\infty B(k)\cos kx\,dk$ for the function $B(k)$ specified in the problem.

SET UP: $\int \cos kx\,dk=\frac{1}{x}\sin kx.$

EXECUTE: **(a)** $\psi(x)=\int_0^\infty B(k)\cos kx dk=\int_0^{k_0}\left(\frac{1}{k_0}\right)\cos kx dk=\frac{\sin kx}{k_0x}\bigg|_0^{k_0}=\frac{\sin k_0x}{k_0x}$

(b) $\psi(x)$ has a maximum value at the origin $x=0$. $\psi(x_0)=0$ when $k_0x_0=\pi$ so $x_0=\frac{\pi}{k_0}$. Thus the width of this function $w_x=2x_0=\frac{2\pi}{k_0}$. If $k_0=\frac{2\pi}{L}$, $w_x=L$. $B(k)$ versus k is graphed in Figure 40.49a. The graph of $\psi(x)$ versus x is in Figure 40.49b.

(c) If $k_0=\frac{\pi}{L}$, $w_x=2L$.

EVALUATE: **(d)** $w_pw_x=\left(\frac{hw_k}{2\pi}\right)\left(\frac{2\pi}{k_0}\right)=\frac{hw_k}{k_0}=\frac{hk_0}{k_0}=h$. If $\Delta x=w_x$ and $\Delta p_x=w_p$, then the uncertainty principle states that $w_pw_x\ge\frac{\hbar}{2}$. For us, no matter what k_0 is, $w_pw_x=h$, which is greater than $\hbar/2$.

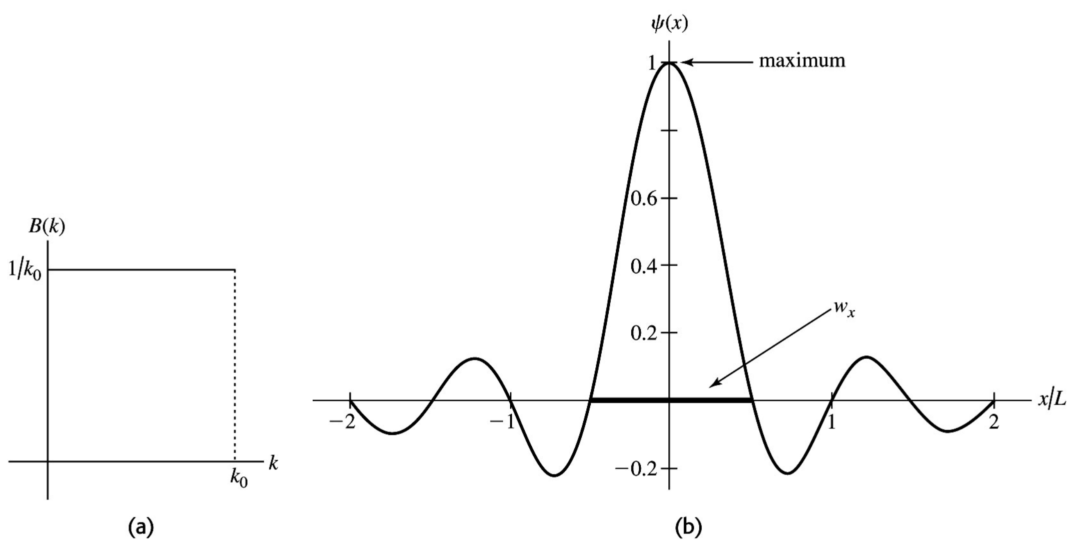

Figure 40.49

40.50. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given function is $\psi(x)=Ae^{ikx}$, and the one-dimensional Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x).$$

EXECUTE: Start with the given function and take the indicated derivatives: $\psi(x)=Ae^{ikx}$.

$$\frac{d\psi(x)}{dx}=Aike^{ikx}.\quad \frac{d^2\psi(x)}{dx^2}=Ai^2k^2e^{ikx}=-Ak^2e^{ikx}.\quad \frac{d^2\psi(x)}{dx^2}=-k^2\psi(x).\quad -\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}=\frac{\hbar^2}{2m}k^2\psi(x).$$

Substituting these results into the one-dimensional Schrödinger equation gives

$$\frac{\hbar^2k^2}{2m}\psi(x)+U_0\psi(x)=E\,\psi(x).$$

EVALUATE: $\psi(x) = A\,e^{ikx}$ is a solution to the one-dimensional Schrödinger equation if $E - U_0 = \dfrac{\hbar^2 k^2}{2m}$ or $k = \sqrt{\dfrac{2m(E - U_0)}{\hbar^2}}$. (Since $U_0 < E$ was given, k is the square root of a positive quantity.) In terms of the particle's momentum p: $k = p/\hbar$, and in terms of the particle's de Broglie wavelength λ: $k = 2\pi/\lambda$.

40.51. IDENTIFY: Let I refer to the region $x < 0$ and let II refer to the region $x > 0$, so $\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$ and $\psi_{II}(x) = Ce^{ik_2x}$. Set $\psi_I(0) = \psi_{II}(0)$ and $\dfrac{d\psi_I}{dx} = \dfrac{d\psi_{II}}{dx}$ at $x = 0$.

SET UP: $\dfrac{d}{dx}(e^{ikx}) = ike^{ikx}$.

EXECUTE: $\psi_I(0) = \psi_{II}(0)$ gives $A + B = C$. $\dfrac{d\psi_I}{dx} = \dfrac{d\psi_{II}}{dx}$ at $x = 0$ gives $ik_1A - ik_1B = ik_2C$. Solving this pair of equations for B and C gives $B = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)A$ and $C = \left(\dfrac{2k_2}{k_1 + k_2}\right)A$.

EVALUATE: The probability of reflection is $R = \dfrac{B^2}{A^2} = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$. The probability of transmission is $T = \dfrac{C^2}{A^2} = \dfrac{4k_1^2}{(k_1 + k_2)^2}$. Note that $R + T = 1$.

40.52. IDENTIFY: For a particle in a box, $E_n = \dfrac{n^2h^2}{8mL^2}$.

SET UP: $\Delta E_n = E_{n+1} - E_n$

EXECUTE: **(a)** $R_n = \dfrac{(n+1)^2 - n^2}{n^2} = \dfrac{2n+1}{n^2} = \dfrac{2}{n} + \dfrac{1}{n^2}$. This is never larger than it is for $n = 1$, and $R_1 = 3$.

EVALUATE: **(b)** R_n approaches zero as n becomes very large. In the classical limit there is no quantization and the spacing of successive levels is vanishingly small compared to the energy levels. Therefore, R_n for a particle in a box approaches the classical value as n becomes very large.

40.53. IDENTIFY and **SET UP:** The energy levels are given by Eq. (40.31): $E_n = \dfrac{n^2h^2}{8mL^2}$. Calculate ΔE for the transition and set $\Delta E = hc/\lambda$, the energy of the photon.

EXECUTE: **(a)** Ground level, $n = 1$, $E_1 = \dfrac{h^2}{8mL^2}$. First excited level, $n = 2$, $E_2 = \dfrac{4h^2}{8mL^2}$. The transition energy is $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$. Set the transition energy equal to the energy hc/λ of the emitted photon. This gives $\dfrac{hc}{\lambda} = \dfrac{3h^2}{8mL^2}$. $\lambda = \dfrac{8mcL^2}{3h} = \dfrac{8(9.109\times10^{-31}\text{ kg})(2.998\times10^{8}\text{ m/s})(4.18\times10^{-9}\text{ m})^2}{3(6.626\times10^{-34}\text{ J}\cdot\text{s})}$.

$\lambda = 1.92\times10^{-5}\text{ m} = 19.2\ \mu\text{m}$.

(b) Second excited level has $n = 3$ and $E_3 = \dfrac{9h^2}{8mL^2}$ The transition energy is

$\Delta E = E_3 - E_2 = \dfrac{9h^2}{8mL^2} - \dfrac{4h^2}{8mL^2} = \dfrac{5h^2}{8mL^2}$. $\dfrac{hc}{\lambda} = \dfrac{5h^2}{8mL^2}$ so $\lambda = \dfrac{8mcL^2}{5h} = \dfrac{3}{5}(19.2\ \mu m) = 11.5\ \mu\text{m}$.

EVALUATE: The energy spacing between adjacent levels increases with n, and this corresponds to a shorter wavelength and more energetic photon in part (b) than in part (a).

40.54. **IDENTIFY:** The probability of finding the particle between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2\, dx$.

SET UP: For the ground state $\psi_1 = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L}$. $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. $\int \cos\alpha x\, dx = \frac{1}{\alpha}\sin\alpha x$.

EXECUTE: **(a)** $\frac{2}{L}\int_0^{L/4} \sin^2\frac{\pi x}{L} dx = \frac{2}{L}\int_0^{L/4}\frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_0^{L/4} = \frac{1}{4} - \frac{1}{2\pi}$, which is about 0.0908.

(b) Repeating with limits of $L/4$ and $L/2$ gives $\frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_{L/4}^{L/2} = \frac{1}{4} + \frac{1}{2\pi}$, about 0.409.

(c) The particle is much likely to be nearer the middle of the box than the edge.

EVALUATE: **(d)** The results sum to exactly $\frac{1}{2}$. Since the probability of the particle being anywhere in the box is unity, the probability of the particle being found between $x = L/2$ and $x = L$ is also $\frac{1}{2}$. This means that the particle is as likely to be between $x = 0$ and $L/2$ as it is to be between $x = L/2$ and $x = L$.

(e) These results are consistent with Figure 40.12b in the textbook. This figure shows a greater probability near the center of the box. It also shows symmetry of $|\psi|^2$ about the center of the box.

40.55. **IDENTIFY:** The probability of the particle being between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2\, dx$, where ψ is the normalized wave function for the particle.

(a) SET UP: The normalized wave function for the ground state is $\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$.

EXECUTE: The probability P of the particle being between $x = L/4$ and $x = 3L/4$ is

$P = \int_{L/4}^{3L/4} |\psi_1|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{\pi x}{L}\right)dx$. Let $y = \pi x/L;\ dx = (L/\pi)\,dy$ and the integration limits become $\pi/4$ and $3\pi/4$.

$$P = \frac{2}{L}\left(\frac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4}\sin^2 y\, dy = \frac{2}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/4}^{3\pi/4}$$

$$P = \frac{2}{\pi}\left[\frac{3\pi}{8} - \frac{\pi}{8} - \frac{1}{4}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right]$$

$P = \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{1}{4}(-1) + \frac{1}{4}(1)\right) = \frac{1}{2} + \frac{1}{\pi} = 0.818$. (Note: The integral formula $\int \sin^2 y\, dy = \frac{1}{2}y - \frac{1}{4}\sin 2y$ was used.)

(b) SET UP: The normalized wave function for the first excited state is $\psi_2 = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$.

EXECUTE: $P = \int_{L/4}^{3L/4}|\psi_2|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{2\pi x}{L}\right)dx$. Let $y = 2\pi x/L;\ dx = (L/2\pi)\,dy$ and the integration limits become $\pi/2$ and $3\pi/2$.

$$P = \frac{2}{L}\left(\frac{L}{2\pi}\right)\int_{\pi/2}^{3\pi/2}\sin^2 y\, dy = \frac{1}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/2}^{3\pi/2} = \frac{1}{\pi}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = 0.500$$

(c) EVALUATE: These results are consistent with Figure 40.11b in the textbook. That figure shows that $|\psi|^2$ is more concentrated near the center of the box for the ground state than for the first excited state; this is consistent with the answer to part (a) being larger than the answer to part (b). Also, this figure shows that for the first excited state half the area under $|\psi|^2$ curve lies between $L/4$ and $3L/4$, consistent with our answer to part (b).

40.56. **IDENTIFY:** The probability is $|\psi|^2\,dx$, with ψ evaluated at the specified value of x.

SET UP: For the ground state, the normalized wave function is $\psi_1 = \sqrt{2/L}\,\sin(\pi x/L)$.

EXECUTE: **(a)** $(2/L)\sin^2(\pi/4)dx = dx/L$.

(b) $(2/L)\sin^2(\pi/2)dx = 2dx/L$

(c) $(2/L)\sin^2(3\pi/4) = dx/L$

EVALUATE: Our results agree with Figure 40.12b in the textbook. $|\psi|^2$ is largest at the center of the box, at $x = L/2$. $|\psi|^2$ is symmetric about the center of the box, so is the same at $x = L/4$ as at $x = 3L/4$.

40.57. **IDENTIFY** and **SET UP:** The normalized wave function for the $n = 2$ first excited level is $\psi_2 = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$. $P = |\psi(x)|^2\,dx$ is the probability that the particle will be found in the interval x to $x + dx$.

EXECUTE: **(a)** $x = L/4$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{L}}.$$

$P = (2/L)dx$

(b) $x = L/2$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{2}\right)\right) = \sqrt{\frac{2}{L}}\sin(\pi) = 0.$$

$P = 0$

(c) $x = 3L/4$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{3L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{2}\right) = -\sqrt{\frac{2}{L}}.$$

$P = (2/L)dx$

EVALUATE: Our results are consistent with the $n = 2$ part of Figure 40.12 in the textbook. $|\psi|^2$ is zero at the center of the box and is symmetric about this point.

40.58. **IDENTIFY:** The impulse applied to a particle equals its change in momentum.

SET UP: For a particle in a box, the magnitude of its momentum is $p = \hbar k = \dfrac{nh}{2L}$ (Eq. 40.29).

EXECUTE: $\Delta\vec{p} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}}$. $|\vec{p}| = \hbar k = \dfrac{\hbar n\pi}{L} = \dfrac{hn}{2L}$. At $x = 0$ the initial momentum at the wall is $\vec{p}_{\text{initial}} = -\dfrac{hn}{2L}\hat{i}$ and the final momentum, after turning around, is $\vec{p}_{\text{final}} = +\dfrac{hn}{2L}\hat{i}$. So, $\Delta\vec{p} = +\dfrac{hn}{2L}\hat{i} - \left(-\dfrac{hn}{2L}\hat{i}\right) = +\dfrac{hn}{L}\hat{i}$. At $x = L$ the initial momentum is $\vec{p}_{\text{initial}} = +\dfrac{hn}{2L}\hat{i}$ and the final momentum, after turning around, is $\vec{p}_{\text{final}} = -\dfrac{hn}{2L}\hat{i}$. So, $\Delta\vec{p} = -\dfrac{hn}{2L}\hat{i} - \dfrac{hn}{2L}\hat{i} = -\dfrac{hn}{L}\hat{i}$.

EVALUATE: The impulse increases with n.

40.59. **IDENTIFY:** Carry out the calculations that are specified in the problem.

SET UP: For a free particle, $U(x) = 0$ so Schrödinger's equation becomes $\dfrac{d^2\psi(x)}{dx^2} = -\dfrac{2m}{\hbar^2}E\psi(x)$.

EXECUTE: **(a)** The graph is given in Figure 40.59.

(b) For $x < 0$: $\psi(x) = e^{+\kappa x}$. $\dfrac{d\psi(x)}{dx} = \kappa e^{+\kappa x}$. $\dfrac{d^2\psi(x)}{dx} = \kappa^2 e^{+\kappa x}$. So $\kappa^2 = -\dfrac{2m}{\hbar^2}E \Rightarrow E = -\dfrac{\hbar^2\kappa^2}{2m}$.

(c) For $x > 0$: $\psi(x) = e^{-\kappa x}$. $\dfrac{d\psi(x)}{dx} = -\kappa e^{-\kappa x}$. $\dfrac{d^2\psi(x)}{dx} = \kappa^2 e^{-\kappa x}$. So again $\kappa^2 = -\dfrac{2m}{\hbar^2}E \Rightarrow E = \dfrac{-\hbar^2\kappa^2}{2m}$.

Parts (b) and (c) show $\psi(x)$ satisfies the Schrödinger's equation, provided $E = \dfrac{-\hbar^2\kappa^2}{2m}$.

EVALUATE: (d) $\dfrac{d\psi(x)}{dx}$ is discontinuous at $x = 0$. (That is, it is negative for $x > 0$ and positive for $x < 0$.) Therefore, this ψ is not an acceptable wave function; $d\psi/dx$ must be continuous everywhere, except where $U \to \infty$.

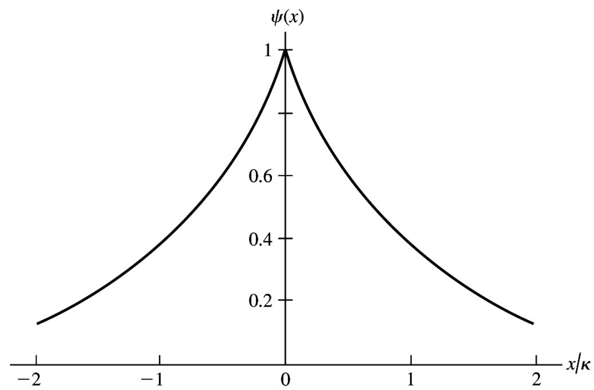

Figure 40.59

40.60. **IDENTIFY:** We start with the penetration distance formula given in the problem.

SET UP: The given formula is $\eta = \dfrac{\hbar}{\sqrt{2m(U_0 - E)}}$.

EXECUTE: (a) Substitute the given numbers into the formula:

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055 \times 10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(9.11 \times 10^{-31}\ \text{kg})(20\ \text{eV} - 13\ \text{eV})(1.602 \times 10^{-19}\ \text{J/eV})}} = 7.4 \times 10^{-11}\ \text{m}$$

(b) $$\eta = \frac{1.055 \times 10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(1.67 \times 10^{-27}\ \text{kg})(30\ \text{MeV} - 20\ \text{MeV})(1.602 \times 10^{-13}\ \text{J/MeV})}} = 1.44 \times 10^{-15}\ \text{m}$$

EVALUATE: The penetration depth varies widely depending on the mass and energy of the particle.

40.61. **IDENTIFY:** Eq. (40.38) applies for $0 \le x \le L$. Eq. (40.40) applies for $x < 0$ and $x > L$. $D = 0$ for $x < 0$ and $C = 0$ for $x > L$.

SET UP: Let $k = \dfrac{\sqrt{2mE}}{\hbar}$. $\dfrac{d}{dx}\sin kx = k\cos kx$. $\dfrac{d}{dx}\cos kx = -k\sin kx$. $\dfrac{d}{dx}e^{\kappa x} = \kappa e^{\kappa x}$. $\dfrac{d}{dx}e^{-\kappa x} = -\kappa e^{-\kappa x}$.

EXECUTE: (a) We set the solutions for inside and outside the well equal to each other at the well boundaries, $x = 0$ and L.

$x = 0$: $B\sin(0) + A = C \Rightarrow A = C$, since we must have $D = 0$ for $x < 0$.

$x = L$: $B\sin\dfrac{\sqrt{2mE}L}{\hbar} + A\cos\dfrac{\sqrt{2mE}L}{\hbar} = +De^{-\kappa L}$ since $C = 0$ for $x > L$.

This gives $B\sin kL + A\cos kL = De^{-\kappa L}$, where $k = \dfrac{\sqrt{2mE}}{\hbar}$.

(b) Requiring continuous derivatives at the boundaries yields

$x = 0$: $\dfrac{d\psi}{dx} = kB\cos(k\cdot 0) - kA\sin(k\cdot 0) = kB = \kappa Ce^{k\cdot 0} \Rightarrow kB = \kappa C$.

$x = L$: $kB\cos kL - kA\sin kL = -\kappa De^{-\kappa L}$

EVALUATE: These boundary conditions allow for B, C, and D to be expressed in terms of an overall normalization constant A.

40.62. **IDENTIFY:** $T = Ge^{-2\kappa L}$ with $G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)$ and $\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$, so $L = -\frac{1}{2\kappa}\ln\left(\frac{T}{G}\right)$.

SET UP: $E = 5.5\text{ eV}, U_0 = 10.0\text{ eV}, m = 9.11\times 10^{-31}\text{ kg}$, and $T = 0.0010$.

EXECUTE: $\kappa = \frac{\sqrt{2(9.11\times 10^{-31}\text{ kg})(4.5\text{ eV})(1.60\times 10^{-19}\text{ J/eV})}}{(1.054\times 10^{-34}\text{ J}\cdot\text{s})} = 1.09\times 10^{10}\text{ m}^{-1}$

and $G = 16\frac{5.5\text{ eV}}{10.0\text{ eV}}\left(1-\frac{5.5\text{ eV}}{10.0\text{ eV}}\right) = 3.96$,

so $L = -\frac{1}{2(1.09\times 10^{10}\text{ m}^{-1})}\ln\left(\frac{0.0010}{3.96}\right) = 3.8\times 10^{-10}\text{ m} = 0.38\text{ nm}$.

EVALUATE: The energies here are comparable to those of electrons in atoms, and the barrier width we calculated is on the order of the diameter of an atom.

40.63. **IDENTIFY** and **SET UP:** When κL is large, then $e^{\kappa L}$ is large and $e^{-\kappa L}$ is small. When κL is small, $\sinh \kappa L \to \kappa L$. Consider both κL large and κL small limits.

EXECUTE: **(a)** $T = \left[1 + \frac{(U_0 \sinh \kappa L)^2}{4E(U_0 - E)}\right]^{-1}$

$\sinh \kappa L = \frac{e^{\kappa L} - e^{-\kappa L}}{2}$

For $\kappa L \gg 1$, $\sinh \kappa L \to \frac{e^{\kappa L}}{2}$ and $T \to \left[1 + \frac{U_0^2 e^{2\kappa L}}{16E(U_0 - E)}\right]^{-1} = \frac{16E(U_0 - E)}{16E(U_0 - E) + U_0^2 e^{2\kappa L}}$

For $\kappa L \gg 1$, $16E(U_0 - E) + U_0^2 e^{2\kappa L} \to U_0^2 e^{2\kappa L}$

$T \to \frac{16E(U_0 - E)}{U_0^2 e^{2\kappa L}} = 16\left(\frac{E}{U_0}\right)\left(1 - \frac{E}{U_0}\right)e^{-2\kappa L}$, which is Eq. (40.42).

(b) $\kappa L = \frac{L\sqrt{2m(U_0 - E)}}{\hbar}$. So $\kappa L \gg 1$ when L is large (barrier is wide) or $U_0 - E$ is large. (E is small compared to U_0.)

(c) $\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$; κ becomes small as E approaches U_0. For κ small, $\sinh \kappa L \to \kappa L$ and

$T \to \left[1 + \frac{U_0^2 \kappa^2 L^2}{4E(U_0 - E)}\right]^{-1} = \left[1 + \frac{U_0^2 2m(U_0 - E)L^2}{\hbar^2 4E(U_0 - E)}\right]^{-1}$ (using the definition of κ).

Thus $T \to \left[1 + \frac{2U_0^2 L^2 m}{4E\hbar^2}\right]^{-1}$

$U_0 \to E$ so $\frac{U_0^2}{E} \to E$ and $T \to \left[1 + \frac{2EL^2 m}{4\hbar^2}\right]^{-1}$

But $k^2 = \frac{2mE}{\hbar^2}$, so $T \to \left[1 + \left(\frac{kL}{2}\right)^2\right]^{-1}$, as was to be shown.

EVALUATE: When κL is large Eq. (40.41) applies and T is small. When $E \to U_0$, T does not approach unity.

40.64. **IDENTIFY:** Compare the energy E of the oscillator to Eq. (40.46) in order to determine n.

SET UP: At the equilibrium position the potential energy is zero and the kinetic energy equals the total energy.

EXECUTE: **(a)** $E=\frac{1}{2}mv^2=[n+(1/2)]\hbar\omega=[n+(1/2)]hf$, and solving for n,

$$n=\frac{\frac{1}{2}mv^2}{hf}-\frac{1}{2}=\frac{(1/2)(0.020\text{ kg})(0.360\text{ m/s})^2}{(6.63\times10^{-34}\text{ J}\cdot\text{s})(1.50\text{ Hz})}-\frac{1}{2}=1.3\times10^{30}.$$

(b) The difference between energies is $\hbar\omega=hf=(6.63\times10^{-34}\text{ J}\cdot\text{s})(1.50\text{ Hz})=9.95\times10^{-34}\text{ J}$. This energy is too small to be detected with current technology.

EVALUATE: This oscillator can be described classically; quantum effects play no measurable role.

40.65. **IDENTIFY** and **SET UP:** Calculate the angular frequency ω of the pendulum and apply Eq. (40.46) for the energy levels.

EXECUTE: $\omega=\frac{2\pi}{T}=\frac{2\pi}{0.500\text{ s}}=4\pi\text{ s}^{-1}$

The ground-state energy is $E_0=\frac{1}{2}\hbar\omega=\frac{1}{2}(1.055\times10^{-34}\text{ J}\cdot\text{s})(4\pi\text{ s}^{-1})=6.63\times10^{-34}\text{ J}$.

$$E_0=6.63\times10^{-34}\text{ J}(1\text{ eV}/1.602\times10^{-19}\text{ J})=4.14\times10^{-15}\text{ eV}$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega$$

$$E_{n+1}=\left(n+1+\frac{1}{2}\right)\hbar\omega$$

The energy difference between the adjacent energy levels is

$\Delta E=E_{n+1}-E_n=\hbar\omega=2E_0=1.33\times10^{-33}\text{ J}=8.30\times10^{-15}\text{ eV}.$

EVALUATE: These energies are much too small to detect. Quantum effects are not important for ordinary size objects.

40.66. **IDENTIFY:** We model the electrons in the lattice as a particle in a box. The energy of the photon is equal to the energy difference between the two energy states in the box.

SET UP: The energy of an electron in the n^{th} level is $E_n=\frac{n^2h^2}{8mL^2}$. We do not know the initial or final levels, but we do know they differ by 1. The energy of the photon, hc/λ, is equal to the energy difference between the two states.

EXECUTE: The energy difference between the levels is $\Delta E=\frac{hc}{\lambda}=\frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{1.649\times10^{-7}\text{ m}}=$ $1.206\times10^{-18}\text{ J}$. Using the formula for the energy levels in a box, this energy difference is equal to

$$\Delta E=\left[n^2-(n-1)^2\right]\frac{h^2}{8mL^2}=(2n-1)\frac{h^2}{8mL^2}.$$

Solving for n gives $n=\frac{1}{2}\left(\frac{\Delta E8mL^2}{h^2}+1\right)=\frac{1}{2}\left(\frac{(1.206\times10^{-18}\text{ J})8(9.11\times10^{-31}\text{ kg})(0.500\times10^{-9}\text{ m})^2}{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}+1\right)=3.$

The transition is from $n=3$ to $n=2$.

EVALUATE: We know the transition is not from the $n=4$ to the $n=3$ state because we let n be the higher state and $n-1$ the lower state.

40.67. **IDENTIFY:** At a maximum, the derivative of the probability function is zero.

SET UP and **EXECUTE:** $\psi(x)=Ce^{-\alpha x^2}$, where $\alpha=\frac{\sqrt{mk'}}{2\hbar}$. $|\psi(x)|^2=|C|^2e^{-2\alpha x^2}$. At values of x where $|\psi(x)|^2$ is a maximum, $\frac{d|\psi(x)|^2}{dx}=0$ and $\frac{d^2|\psi(x)|^2}{dx^2}<0$. $\frac{d|\psi(x)|^2}{dx}=|C|^2(-2\alpha x)e^{-2\alpha x^2}=0$. Only solution is $x=0$. $\frac{d^2|\psi(x)|^2}{dx^2}=|C|^2\left[-2\alpha e^{-2\alpha x^2}+4\alpha^2 x\, e^{-2\alpha x^2}\right]$. At $x=0$, $\frac{d^2|\psi(x)|^2}{dx^2}=|C|^2(-2\alpha)<0$, so $|\psi(x)|^2$ is a maximum at $x=0$.

EVALUATE: There is only one maximum, at $x=0$, so the probability function peaks only there.

40.68. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given wave function is $\psi_1(x)=A_1 x e^{-\alpha^2x^2/2}$ and the Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\,\psi(x).$$

EXECUTE: **(a)** Start by taking the indicated derivatives: $\psi_1(x)=A_1xe^{-\alpha^2x^2/2}$.

$$\frac{d\psi_1(x)}{dx}=-\alpha^2x^2A_1e^{-\alpha^2x^2/2}+A_1e^{-\alpha^2x^2/2}.$$

$$\frac{d^2\psi_1(x)}{dx^2}=-A_1\alpha^2 2xe^{-\alpha^2x^2/2}-A_1\alpha^2x^2(-\alpha^2x)e^{-\alpha^2x^2/2}+A_1(-\alpha^2x)e^{-\alpha^2x^2/2}.$$

$$\frac{d^2\psi_1(x)}{dx^2}=\left[-2\alpha^2+(\alpha^2)^2x^2-\alpha^2\right]\psi_1(x)=\left[-3\alpha^2+(\alpha^2)^2x^2\right]\psi_1(x).$$

$$-\frac{\hbar}{2m}\frac{d^2\psi_1(x)}{dx^2}=-\frac{\hbar^2}{2m}\left[-3\alpha^2+(\alpha^2)^2x^2\right]\psi_1(x).$$

Equation (40.44) is $-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\,\psi(x)$. Substituting the above result into that equation gives $-\frac{\hbar^2}{2m}\left[-3\alpha^2+(\alpha^2)^2x^2\right]\psi_1(x)+\frac{k'x^2}{2}\psi_1(x)=E\,\psi_1(x)$. Since $\alpha^2=\frac{m\omega}{\hbar}$ and $\omega=\sqrt{\frac{k'}{m}}$, the coefficient of x^2 is $-\frac{\hbar^2}{2m}(\alpha^2)^2+\frac{k'}{2}=-\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar}\right)^2+\frac{m\omega^2}{2}=0$.

(b) $A_1=\left(\frac{m\omega}{\hbar}\right)^{3/4}\left(\frac{4}{\pi}\right)^{1/4}$

(c) The probability density function $|\psi|^2$ is $|\psi_1(x)|^2=A_1^2x^2e^{-\alpha^2x^2}$

At $x=0$, $|\psi_1|^2=0$. $\frac{d|\psi_1(x)|^2}{dx}=A_1^2 2xe^{-\alpha^2x^2}+A_1^2x^2(-\alpha^2 2x)e^{-\alpha^2x^2}=A_1^2 2xe^{-\alpha^2x^2}-A_1^2 2x^3\alpha^2e^{-\alpha^2x^2}$.

At $x=0$, $\frac{d|\psi_1(x)|^2}{dx}=0$. At $x=\pm\frac{1}{\alpha}$, $\frac{d|\psi_1(x)|^2}{dx}=0$.

$$\frac{d^2|\psi_1(x)|^2}{dx^2}=A_1^2 2e^{-\alpha^2x^2}+A_1^2 2x(-\alpha^2 2x)e^{-\alpha^2x^2}-A_1^2 2(3x^2)\alpha^2e^{-\alpha^2x^2}-A_1^2 2x^3\alpha^2(-\alpha^2 2x)e^{-\alpha^2x^2}.$$

$\frac{d^2|\psi_1(x)|^2}{dx^2}=A_1^2 2e^{-\alpha^2x^2}-A_1^2 4x^2\alpha^2e^{-\alpha^2x^2}-A_1^2 6x^2\alpha^2e^{-\alpha^2x^2}+A_1^2 8x^4(\alpha^2)^2e^{-\alpha^2x^2}$. At $x=0$,

$\frac{d^2|\psi_1(x)|^2}{dx^2}>0$. So at $x=0$, the first derivative is zero and the second derivative is positive. Therefore, the probability density function has a minimum at $x=0$. At $x=\pm\frac{1}{\alpha}$, $\frac{d^2|\psi_1(x)|^2}{dx^2}<0$. So at $x=\pm\frac{1}{\alpha}$, the first derivative is zero and the second derivative is negative. Therefore, the probability density function has maxima at $x=\pm\frac{1}{\alpha}$, corresponding to the classical turning points for $n=0$ as found in the previous question.

EVALUATE: $\psi_1(x) = A_1 x e^{-\alpha^2 x^{2/2}}$ is a solution to Eq. (40.44) if $-\dfrac{\hbar^2}{2m}(-3\alpha^2)\psi_1(x) = E\,\psi_1(x)$ or $E = \dfrac{3\hbar^2\alpha^2}{2m} = \dfrac{3\hbar\omega}{2}$. $E_1 = \dfrac{3\hbar\omega}{2}$ corresponds to $n = 1$ in Equation (40.46).

40.69. **IDENTIFY:** For a standing wave in the box, there must be a node at each wall and $n\left(\dfrac{\lambda}{2}\right) = L$.

SET UP: $p = \dfrac{h}{\lambda}$ so $mv = \dfrac{h}{\lambda}$.

EXECUTE: **(a)** For a standing wave, $n\lambda = 2L$, and $E_n = \dfrac{p^2}{2m} = \dfrac{(h/\lambda)^2}{2m} = \dfrac{n^2h^2}{8mL^2}$.

(b) With $L = a_0 = 0.5292\times10^{-10}$ m, $E_1 = 2.15\times10^{-17}$ J $= 134$ eV.

EVALUATE: For a hydrogen atom, E_n is proportional to $1/n^2$ so this is a very poor model for a hydrogen atom. In particular, it gives very inaccurate values for the separations between energy levels.

40.70. **IDENTIFY** and **SET UP:** Follow the steps specified in the problem.

EXECUTE: **(a)** As with the particle in a box, $\psi(x) = A\sin kx$, where A is a constant and $k^2 = 2mE/\hbar^2$. Unlike the particle in a box, however, k and hence E do not have simple forms.

(b) For $x > L$, the wave function must have the form of Eq. (40.40). For the wave function to remain finite as $x \to \infty$, $C = 0$. The constant $\kappa^2 = 2m(U_0 - E)/\hbar$, as in Eq. (40.40).

(c) At $x = L$, $A\sin kL = De^{-\kappa L}$ and $kA\cos kL = -\kappa De^{-\kappa L}$. Dividing the second of these by the first gives $k\cot kL = -\kappa$, a transcendental equation that must be solved numerically for different values of the length L and the ratio E/U_0.

EVALUATE: When $U_0 \to \infty$, $\kappa \to \infty$ and $\dfrac{\cos(kL)}{\sin(kL)} \to \infty$. The solutions become $k = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots,$ the same as for a particle in a box.

40.71. **IDENTIFY:** Require $\psi(-L/2) = \psi(L/2) = 0$.

SET UP: $k = \dfrac{2\pi}{\lambda}$, $p = \dfrac{h}{\lambda}$ and $E = \dfrac{p^2}{2m}$.

EXECUTE: **(a)** $\psi(x) = A\sin kx$ and $\psi(-L/2) = 0 = \psi(+L/2)$

$$\Rightarrow 0 = A\sin\left(\frac{+kL}{2}\right) \Rightarrow \frac{+kL}{2} = n\pi \Rightarrow k = \frac{2n\pi}{L} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{L}{n} \Rightarrow p_n = \frac{h}{\lambda} = \frac{nh}{L} \Rightarrow E_n = \frac{p^2}{2m} = \frac{n^2h^2}{2mL^2} = \frac{(2n)^2h^2}{8mL^2}, \text{ where } n = 1, 2\ldots$$

(b) $\psi(x) = A\cos kx$ and $\psi(-L/2) = 0 = \psi(+L/2)$

$$\Rightarrow 0 = A\cos\left(\frac{kL}{2}\right) \Rightarrow \frac{kL}{2} = (2n+1)\frac{\pi}{2} \Rightarrow k = \frac{(2n+1)\pi}{L} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{2L}{(2n+1)} \Rightarrow p_n = \frac{(2n+1)h}{2L}$$

$$\Rightarrow E_n = \frac{(2n+1)^2h^2}{8mL^2}\; n = 0, 1, 2\ldots$$

(c) The combination of all the energies in parts (a) and (b) is the same energy levels as given in Eq. (40.31), where $E_n = \dfrac{n^2h^2}{8mL^2}$.

EVALUATE: **(d)** Part (a)'s wave functions are odd, and part (b)'s are even.

40.72. **IDENTIFY** and **SET UP:** Follow the steps specified in the problem.

EXECUTE: **(a)** $E = K + U(x) = \dfrac{p^2}{2m} + U(x) \Rightarrow p = \sqrt{2m(E - U(x))}$. $\lambda = \dfrac{h}{p} \Rightarrow \lambda(x) = \dfrac{h}{\sqrt{2m(E - U(x))}}$.

(b) As $U(x)$ gets larger (i.e., $U(x)$ approaches E from below—recall $k \geq 0$), $E - U(x)$ gets smaller, so $\lambda(x)$ gets larger.

(c) When $E = U(x)$, $E - U(x) = 0$, so $\lambda(x) \to \infty$.

(d) $\displaystyle\int_a^b \frac{dx}{\lambda(x)} = \int_a^b \frac{dx}{h/\sqrt{2m(E-U(x))}} = \frac{1}{h}\int_a^b \sqrt{2m(E-U(x))}\,dx = \frac{n}{2} \Rightarrow \int_a^b \sqrt{2m(E-U(x))}\,dx = \frac{hn}{2}$.

(e) $U(x) = 0$ for $0 < x < L$ with classical turning points at $x = 0$ and $x = L$. So,

$\displaystyle\int_a^b \sqrt{2m(E-U(x))}\,dx = \int_0^L \sqrt{2mE}\,dx = \sqrt{2mE}\int_0^L dx = \sqrt{2mE}\,L$. So, from part (d),

$\sqrt{2mE}\,L = \dfrac{hn}{2} \Rightarrow E = \dfrac{1}{2m}\left(\dfrac{hn}{2L}\right)^2 = \dfrac{h^2n^2}{8mL^2}$.

EVALUATE: **(f)** Since $U(x) = 0$ in the region between the turning points at $x = 0$ and $x = L$, the result is the *same* as part (e). The height U_0 never enters the calculation. WKB is best used with *smoothly* varying potentials $U(x)$.

40.73. **DENTIFY:** Perform the calculations specified in the problem.

SET UP: $U(x) = \frac{1}{2}k'x^2$.

EXECUTE: **(a)** At the turning points $E = \dfrac{1}{2}k'x_{\text{TP}}^2 \Rightarrow x_{\text{TP}} = \pm\sqrt{\dfrac{2E}{k'}}$.

(b) $\displaystyle\int_{-\sqrt{2E/k'}}^{+\sqrt{2E/k'}} \sqrt{2m\left(E - \frac{1}{2}k'x^2\right)}dx = \frac{nh}{2}$. To evaluate the integral, we want to get it into a form that matches

the standard integral given. $\sqrt{2m\left(E - \dfrac{1}{2}k'x^2\right)} = \sqrt{2mE - mk'x^2} = \sqrt{mk'}\sqrt{\dfrac{2mE}{mk'} - x^2} = \sqrt{mk'}\sqrt{\dfrac{2E}{k'} - x^2}$.

Letting $A^2 = \dfrac{2E}{k'}$, $a = -\sqrt{\dfrac{2E}{k'}}$, *and* $b = +\sqrt{\dfrac{2E}{k'}}$

$$\Rightarrow \sqrt{mk'}\int_a^b \sqrt{A^2 - x^2}\,dx = 2\frac{\sqrt{mk'}}{2}\left[x\sqrt{A^2 - x^2} + A^2 \arcsin\left(\frac{x}{|A|}\right)\right]\Bigg|_0^b$$

$$= \sqrt{mk'}\left[\sqrt{\frac{2E}{k'}}\sqrt{\frac{2E}{k'} - \frac{2E}{k'}} + \frac{2E}{k'}\arcsin\left(\frac{\sqrt{2E/k'}}{\sqrt{2E/k'}}\right)\right] = \sqrt{mk'}\,\frac{2E}{k'}\arcsin(1) = 2E\sqrt{\frac{m}{k'}}\left(\frac{1}{2}\right).$$

Using WKB, this is equal to $\dfrac{hn}{2}$, so $E\sqrt{\dfrac{m}{k'}}\pi = \dfrac{hn}{2}$. Recall $\omega = \sqrt{\dfrac{k'}{m}}$, so $E = \dfrac{h}{2\pi}\omega n = \hbar\omega n$.

EVALUATE: **(c)** We are missing the zero-point-energy offset of $\dfrac{\hbar\omega}{2}\left(\text{recall } E = \hbar\omega\left(n + \dfrac{1}{2}\right)\right)$. It underestimates the energy. However, our approximation isn't bad at all!

40.74. **IDENTIFY** and **SET UP:** Perform the calculations specified in the problem.

EXECUTE: **(a)** At the turning points $E = A|x_{\text{TP}}| \Rightarrow x_{\text{TP}} = \pm\dfrac{E}{A}$.

(b) $\displaystyle\int_{-E/A}^{+E/A} \sqrt{2m(E - A|x|)}dx = 2\int_0^{E/A} \sqrt{2m(E - Ax)}\,dx$. Let $y = 2m(E - Ax) \Rightarrow$

$dy = -2mA\,dx$ when $x = \dfrac{E}{A}$, $y = 0$, and when $x = 0$, $y = 2mE$. So

$2\int_0^{\frac{E}{A}} \sqrt{2m(E-Ax)}\,dx = -\frac{1}{mA}\int_{2mE}^{0} y^{1/2}dy = -\frac{2}{3mA}y^{3/2}\Big|_{2mE}^{0} = \frac{2}{3mA}(2mE)^{3/2}$. Using WKB, this is equal to $\frac{hn}{2}$.

So, $\frac{2}{3mA}(2mE)^{3/2} = \frac{hn}{2} \Rightarrow E = \frac{1}{2m}\left(\frac{3mAh}{4}\right)^{2/3} n^{2/3}$.

EVALUATE: (c) The difference in energy decreases between successive levels. For example:

$$1^{2/3} - 0^{2/3} = 1,\ 2^{2/3} - 1^{2/3} = 0.59,\ 3^{3/2} - 2^{3/2} = 0.49, \ldots$$

- A sharp ∞ step gave ever-increasing level differences $(\sim n^2)$.
- A parabola $(\sim x^2)$ gave evenly spaced levels $(\sim n)$.
- Now, a linear potential $(\sim x)$ gives ever-decreasing level differences $(\sim n^{2/3})$.

Roughly speaking, if the curvature of the potential (~ second derivative) is bigger than that of a parabola, then the level differences will increase. If the curvature is less than a parabola, the differences will decrease.